

Frequency Shift and Mode Coupling in the Nonlinear Dynamics of a Bose Condensed Gas

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We investigate the behavior of large amplitude oscillations of a trapped Bose-condensed gas of alkali atoms at zero temperature, by solving the equations of hydrodynamics for collective modes. Due to the atom-atom interaction, the equations of motion are nonlinear and give rise to significant frequency shift and mode coupling. We provide analytic expressions for the frequency shift, pointing out the crucial role played by the anisotropy of the confining potential. For special values of the anisotropy parameter the mode coupling is particularly strong and the frequency shift becomes large, revealing a peculiar behavior of the Bose-condensed gas. Consequences on the theory of collapse and revival of collective excitations are also discussed.

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I. INTRODUCTION

One of the most important features of an interacting quantum many-body system is its response to external oscillating fields. The collective modes, which are expected to dominate the low frequency response, represent a very effective tool for probing the role of interactions and testing theoretical schemes. For this reason, measurements of collective modes in the trapped gases of alkali atoms [1,2] were carried out soon after the discovery of Bose-Einstein condensation [3]. The remarkable agreement between measured frequencies and theoretical predictions [4-9] is one of the first important achievements in the investigation of these new systems. It provides also a clean *a posteriori* justification of the mean-field scheme, based on the formalism of Gross-Pitaevskii theory [10], which is the starting point of most calculations. This theoretical approach is expected to give indeed an accurate description of the ground state and the excited states of such dilute interacting gases at low temperature. The same scheme, when the atom-atom interaction becomes dominant compared with the zero-point quantum pressure, reduces to the Thomas-Fermi approximation for the ground state and to the equations of nondissipative hydrodynamics for the excited states [5,11,12]. This is particularly useful for discussing the relevant physical properties by means of analytic or semi-analytic results, taking advantage of the simplicity and clarity of the hydrodynamic equations.

The same mean-field approach, which correctly reproduces the frequency of the normal modes of the trapped gas in the linear limit (small oscillations around the ground state), is suitable to investigating also the nonlinear dynamics of the systems [13-20]. The nonlinearity is included in the equations of motion through the mean-field, which is proportional to the condensate density. Thus, measurable effects of nonlinearity could represent further clean signatures of Bose-Einstein condensation. Among them, large amplitude oscillations of the condensate can be easily produced in the trapped gases [1,2]; nonlinear effects are expected to give rise to frequency shift, mode coupling, harmonic generation and stochastic motion.

The purpose of the present work is to derive simple differential equations for large amplitude oscillations within the formalism of hydrodynamics. In particular we will provide analytic formulae for the frequency shift of three collective modes in a generic axially-symmetric trap. These formulae allow us to discuss the important role played by the anisotropy of the trap. We will show that special values of the anisotropy parameter (the ratio of the axial and radial frequencies of the trap) can be associated with strong nonlinear effects even for oscillations of relatively small amplitude. The same analytic results have interesting consequences in the theory of collapse and revival of the condensate.

In the next section we introduce the basic formalism. Then, in section III, we discuss how the collective modes can be driven and analysed numerically. In section IV we perform a small amplitudes expansion and derive analytic solutions for the $m = 0$ and $m = 2$ modes and their frequency shifts. In section V we discuss both the numerical and the analytic results for different traps. An application to the theory of collapse and revival of the the oscillations is given in section VI. The paper will end with a short summary.

II. BASIC FORMALISM

Let us start with the hydrodynamic equations in Thomas-Fermi approximation [5]:

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\mathbf{v}\rho) = 0 \quad (1)$$

$$m \frac{\partial}{\partial t} \mathbf{v} + \nabla \left(V_{ext} + g\rho + \frac{mv^2}{2} \right) = 0. \quad (2)$$

Density and velocity are related to the condensate wave function $\Psi(\mathbf{r}, t)$ through $\rho = |\Psi|^2$ and $\mathbf{v} = \hbar(2mi\rho)^{-1}(\Psi^* \nabla \Psi - \Psi \nabla \Psi^*)$. The density is normalized to the number of particles in the condensate, $N = \int d\mathbf{r} \rho(\mathbf{r})$. The external confining potential has the form $V_{ext}(\mathbf{r}) = (1/2)m \sum_i \omega_i^2 r_i^2$, where $r_i \equiv x, y, z$. The trapping frequencies can depend on time, $\omega_i = \omega_i(t)$, in the presence of an external driving force. Their static values, $\omega_{0i} = \omega_i(0)$, fix the equilibrium configuration of the system. For cylindrically symmetric traps one can define the radial frequency $\omega_\perp \equiv \omega_{0x} = \omega_{0y}$ and the asymmetry parameter $\lambda = \omega_{0z}/\omega_\perp$. The external potential provides also the typical length scale of the system in each direction, $a_{HO}^{(i)} = \sqrt{\hbar/(m\omega_{0i})}$. Since the system is dilute, the atom-atom interaction enters only through the quantity $g = 4\pi\hbar^2 a/m$, where a is the s -wave scattering length.

The equations of nondissipative hydrodynamics are equivalent to the time dependent Gross-Pitaevskii equation for the condensate wave function in the limit $Na/a_{HO} \gg 1$, if the interaction is repulsive ($a > 0$). In this case, the effects of the zero point kinetic energy (quantum pressure) become negligible and the gas is dominated by the balance of the internal and external potential energies. The stationary solution of the hydrodynamic equations is the Thomas-Fermi ground state [5,21]:

$$\rho_0^{TF}(\mathbf{r}) = |\Psi_0^{TF}(\mathbf{r})|^2 = g^{-1}[\mu - V_{ext}(\mathbf{r})] \quad \text{for } \mu \geq V_{ext}(\mathbf{r}) \quad (3)$$

where the chemical potential μ is fixed by the normalization of the density to the number of particle N . The hydrodynamic approach works in an excellent way for the lowest collective modes of the Sodium atoms trapped at MIT [2], where N is of the order of 1 million and more. Conversely, it provides only a semi-quantitative description of the Rubidium gas first trapped at Jila [1], where the number of atoms was smaller (10^3 - 10^4). Even in that case, however, the measured frequencies converge nicely to the hydrodynamic predictions for the largest values of N . Compared with the numerical solution of the Gross-Pitaevskii equation, the hydrodynamic formalism has the advantage of providing analytic results for the dispersion law of the collective modes and for other useful quantities.

As already discussed in our previous paper [17], exact solutions of the hydrodynamic equations can be found in the form

$$\rho(\mathbf{r}, t) = a_x(t)x^2 + a_y(t)y^2 + a_z(t)z^2 + a_0(t) \quad (4)$$

$$\mathbf{v} = \frac{1}{2} \nabla [\alpha_x(t)x^2 + \alpha_y(t)y^2 + \alpha_z(t)z^2], \quad (5)$$

restricted to the region where $\rho \geq 0$. With this choice, equations (1-2) transform into a set of coupled differential

equations for the time-dependent coefficients $a_j(t)$ and $\alpha_j(t)$. One of them is fixed by the conservation of N : $a_0 = -(15N/8\pi)^{2/5}(a_x a_y a_z)^{1/5}$. The equations for the others can be further simplified by introducing the new variables b_i defined by $a_i = -m\omega_{0i}^2(2gb_x b_y b_z b_i^2)^{-1}$. The hydrodynamic equations then yield $\alpha_i = \dot{b}_i/b_i$ and

$$\ddot{b}_i + \omega_i^2 b_i - \omega_{0i}^2/(b_i b_x b_y b_z) = 0, \quad (6)$$

with $i = x, y, z$. These equations describe the time evolution of the widths of the atomic cloud, since the new variables b_i are directly related to the mean square radii and velocities of the system [17]: $b_i^2 \propto \langle r_i^2 \rangle$ and $\dot{b}_i^2 \propto \langle v_i^2 \rangle$. Different derivations of equations (6) and some applications are given in Refs. [15–17]. A variational approach including the zero point quantum pressure, beyond the Thomas-Fermi approximation, has been also presented in Ref. [7]. Note again that the frequencies ω_i , entering the second term of (6), can depend on time and, hence, these equations can be used for describing time varying traps, as well as the expansion of the gas after a sudden switching-off of the confining potential. The set of solutions defined by the scaling transformations (4-5) does not exhaust all possible motions of the trapped gas. For instance, the motion of the center of mass can be included by adding terms linear in x , y or z and other solutions can be found including terms of the form xy , xz or yz . However, they are well suited to study the collective modes of lowest multipolarity and energy, namely the $m = 0$ and $m = 2$ modes, where m is the azimuthal angular momentum in the cylindrically symmetric trap. In the following we will apply equation (6) to these modes in the nonlinear regime.

III. OSCILLATIONS OF A DRIVEN CONDENSATE

At equilibrium one has $b_i = 1$ and $\dot{b}_i = 0$. One can perturb the system by modulating the trap frequencies for a certain time and then let it oscillate freely. Formally, this means that equations (6) have to be solved using a time dependent frequency of the form $\omega_i^2(t) = \omega_{0i}^2[1 + 2f_i(t)]$, where $f_i(t)$ is an appropriate sinusoidal function within a finite time interval, while the factor 2 is introduced for convenience. An example is given in Fig. 1 for the case of a trap with asymmetry parameter $\lambda = \omega_{0z}/\omega_\perp = \sqrt{8}$, as in Jila experiments. Starting from equilibrium, an oscillation is induced by choosing $f_x(t) = \eta \sin \Omega_d t$, $f_y = -f_x$ and $f_z = 0$, with a driving frequency $\Omega_d = \sqrt{2} \omega_\perp$. The driving force is switched-off at $t = 20\omega_\perp^{-1}$. The three curves correspond to the width $b_x(t)$ plotted for different values of the driving strength, η . The free oscillations are undamped, since the theory is restricted to zero temperature and does not include any dissipation. In the small amplitude limit, $\eta \rightarrow 0$, the oscillation coincides with the $m = 2$ normal mode predicted by the linearized hydrodynamic equations [5]. It corresponds to a quadrupole-type

excitation in the xy -plane. For larger amplitudes, the response of the system is slightly shifted in frequency, as can be seen from the figure, and the oscillations are no more purely sinusoidal. This is associated with the occurrence of harmonic generation and mode coupling. In a similar way, one can excite the $m = 0$ modes. The one at low energy corresponds to an in-phase oscillation of the width along x and y and out-of-phase along z . The one at high energy is an in-phase compressional mode along all directions (breathing mode). In all cases, one can solve numerically the equations (6) and extract amplitude and frequency of the excited modes. A simple way consists in doing a best-fit with a sinusoidal function to the appropriate width $b_i(t)$, after the switching-off of the driving force. One can also perform a Fourier analysis of the signal. We used both methods, finding the same results provided the time interval contains at least three or four complete oscillations.

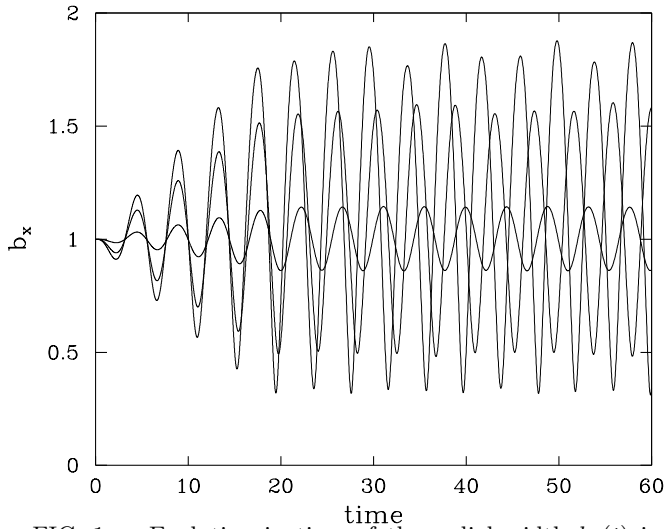


FIG. 1. Evolution in time of the radial width $b_x(t)$ in a trap with $\lambda = \sqrt{8}$. Time is in units of ω_\perp^{-1} . The oscillation is driven by an external force for the first 20 time units; then the condensate oscillates freely. The three curves correspond to increasing the strength of the driving force ($\eta = 0.01, 0.04, 0.06$, see text). In the small amplitude limit, the oscillation coincides with the $m = 2$ normal mode.

In order to point out the effects of mode-coupling, it is convenient to introduce suitable combinations of the widths $b_i(t)$ as follows:

$$b_x(t) = 1 + \xi_-(t) + \xi_+(t) + \xi_2(t) \quad (7)$$

$$b_y(t) = 1 + \xi_-(t) + \xi_+(t) - \xi_2(t) \quad (8)$$

$$b_z(t) = 1 + (q_- - 4) \xi_-(t) + (q_+ - 4) \xi_+(t), \quad (9)$$

with

$$q_\mp = 2 + (3/2)\lambda^2 \mp (1/2)\sqrt{9\lambda^4 - 16\lambda^2 + 16}. \quad (10)$$

Inserting these combinations into (6) one obtains equations for ξ_- , ξ_+ and ξ_2 . Such equations are completely

decoupled in the linear limit, that is, when the perturbations ξ_- , ξ_+ and ξ_2 are much smaller than 1. Thus, these functions represent the normal modes of the system: ξ_2 is the $m = 2$ mode, having frequency $\sqrt{2} \omega_\perp$, while ξ_- and ξ_+ are the low-lying and high-lying $m = 0$ modes, respectively, whose frequencies are $\sqrt{q_\mp} \omega_\perp$ [5]. We note also that $q_+ \neq q_-$ for any value of λ .

In Fig. 2 we plot the functions $\xi_j(t)$ for the oscillation of largest amplitude already shown in Fig. 1. The driving force is tuned on the $m = 2$ mode, ξ_2 , but, due to nonlinear coupling, the low-lying $m = 0$ mode, ξ_- , is also significantly excited, while ξ_+ remains very small. The coupling between these modes causes the irregular oscillation of b_x , given by (7) and plotted in Fig. 1. These oscillations have a very large amplitude (about 80% of the radial width); for smaller oscillations the coupling with ξ_- tends to vanish. For the smallest oscillation in Fig. 1 both ξ_- and ξ_+ are negligible with respect to the driven mode ξ_2 and one has also $b_x(t) \simeq 1 + \xi_2(t)$.

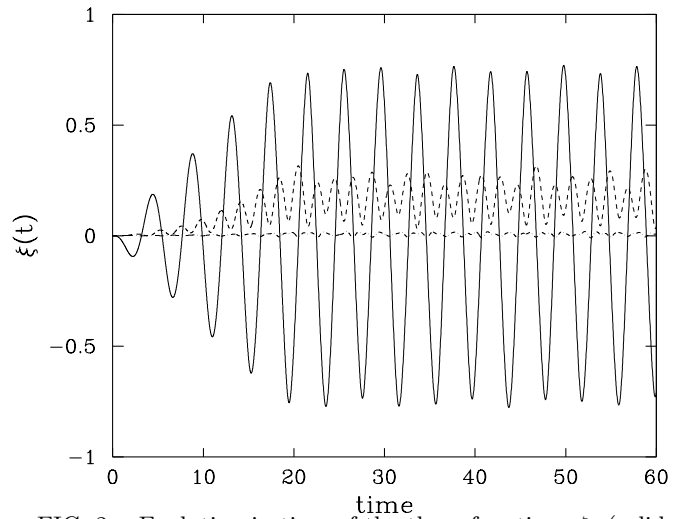


FIG. 2. Evolution in time of the three functions ξ_2 (solid line), ξ_- (dashed line) and ξ_+ (dot-dashed line) in a trap with $\lambda = \sqrt{8}$. Time is in units of ω_\perp^{-1} . In the limit of small amplitude, these functions coincide with the $m = 2$, low-lying $m = 0$ and high-lying $m = 0$ normal modes, respectively. The driving force is the same as for the largest oscillation in Fig. 1.

A strong enhancement of nonlinear effects can be obtained not only by increasing the strength of the driving force, but also by changing the anisotropy of the trap. In fact, for special values of $\lambda = \omega_z/\omega_\perp$, frequencies of different modes, or of their harmonics, can coincide. An example is shown in Fig. 3, where we plot the functions $\xi_j(t)$ for an oscillation driven as in the case of the smallest oscillation in Fig. 1. The driving force is again tuned on ξ_2 with a relatively small strength, but now the oscillation exhibits a strong nonlinear behavior. The crucial parameter is the value of λ , which in Fig. 3 is chosen to be $\sqrt{16/7}$. As we will discuss in the next sections, this special value of λ is associated with a resonance of the two modes $\xi_2(t)$ and $\xi_+(t)$; indeed, the figure shows an

evident beating of these two modes.

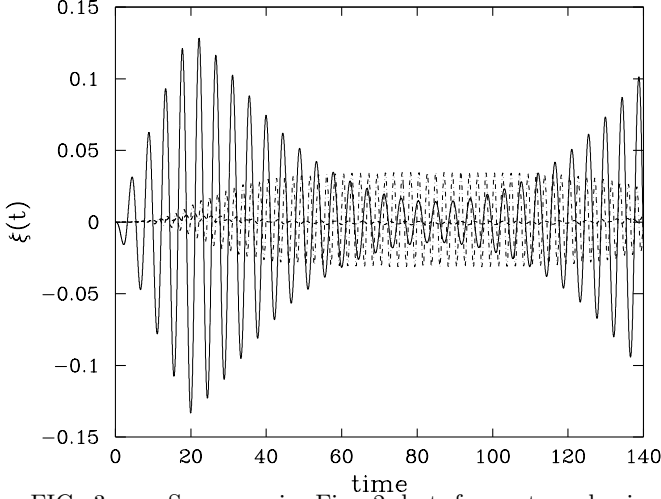


FIG. 3. Same as in Fig. 2 but for a trap having $\lambda = \sqrt{16/7}$. The beating is between the ξ_2 mode, which is excited by the external force, and the ξ_+ mode, which is coupled through the second harmonic of ξ_2 . The driving force is the same as for the smallest oscillation in Fig. 1.

At this point, before presenting further numerical results obtained from equations (6), we prefer to explore the first nonlinear corrections to the uncoupled normal modes. As we will see, an expansion of (6) at small amplitudes provides useful analytic formulae, which contain the main physical ideas and give accurate results.

IV. ANALYTIC EXPANSIONS AT SMALL AMPLITUDE

Equations (6), with $\omega_i = \omega_{0i}$ and $i = x, y, z$, describe free oscillations in the trap. They can be expanded by assuming $b_i(t) = 1 + \delta b_i(t)$ and keeping the lowest orders in the small perturbation δb_i . The linear terms provide the hydrodynamic spectrum of normal modes, while harmonic generation is included in the higher order terms. One can also use definitions (7-9) in order to get three coupled equations for the functions ξ_- , ξ_+ and ξ_2 , up to quadratic or cubic terms. We show here an example of an explicit calculation for the $m = 2$ mode, while for the $m = 0$ modes we will present only the final results.

Let us assume the $m = 2$ mode be excited in the cylindrically symmetric trap by some suitable driving force, in such a way that $\xi_2(t) \approx A \cos(\sqrt{2} \omega_\perp t)$. If $A \ll 1$, the coupling with the other modes is small and, hence, the $m = 0$ oscillations ξ_- and ξ_+ are expected to be of order A^2 . Then, by expanding (6-9) up to third order in A , one gets the following equations:

$$\ddot{\xi}_2 + \omega_\perp^2 \xi_2 [2 - \xi_- q_- - \xi_+ q_+ + 2\xi_2^2] = 0 \quad (11)$$

$$\ddot{\xi}_\mp + \omega_\perp^2 q_\mp \xi_\mp \mp 5\omega_\perp^2 \frac{\lambda^2}{q_\pm} \left(\frac{q_\pm - 4}{q_+ - q_-} \right) \xi_2^2 = 0 \quad (12)$$

The two $m = 0$ modes are not excited directly, but they can be driven by the $m = 2$ mode *via* the last term in (12). One can easily find conditions for resonances: the frequency of the $m = 0$ mode, $\sqrt{q_\mp} \omega_\perp$, should be equal to the frequency of the second harmonic of the $m = 2$ mode, $2\sqrt{2} \omega_\perp$. This never occurs for the lowest $m = 0$ mode, since q_- is always smaller than 10/3. Conversely, for the high-lying mode ξ_+ , a resonance is obtained for $\lambda = \sqrt{16/7}$. For this value of the asymmetry parameter, a small driving force, appropriate for exciting the $m = 2$ mode, give rise to a coupled motion of both the $m = 2$ and high-lying $m = 0$ modes. This is exactly what we have already shown in Fig. 3 by solving the hydrodynamic equations (6) for the same special value of λ . This enhancement of nonlinearity, due to resonances *via* second harmonics, has been recently suggested also in Ref. [22]. Even though equations (11-12) are valid in the limit of small amplitudes, they give the correct conditions for the occurrence of strong nonlinear effects in the *exact* hydrodynamic solutions.

The above equations can be also used to predict the frequency shift of the collective modes at the lowest order in the amplitude A . In fact, equations (12) can be solved explicitly, by inserting ξ_2 at the leading order: $\xi_2(t) = A_2 \cos(\sqrt{2} \omega_\perp t)$. One finds

$$\xi_\mp(t) = \frac{\pm(q_\pm - 4)}{2(q_+ - q_-)(q_\mp - 8)} [q_\mp \xi_2^2(t) - 4A_2^2]. \quad (13)$$

Using these solutions into (11), one gets an equation for ξ_2 at the next order:

$$\begin{aligned} \ddot{\xi}_2 + 2\omega_\perp^2 \left(1 - A_2^2 \frac{16 - 5\lambda^2}{16 - 7\lambda^2} \right) \xi_2 \\ + 2\omega_\perp^2 \left(1 + \frac{16 - 3\lambda^2}{16 - 7\lambda^2} \right) \xi_2^3 = 0. \end{aligned} \quad (14)$$

This is a special case of the more general class of equations for anharmonic oscillators:

$$\ddot{\xi} + \Omega_0^2 f + \gamma_2 \xi^2 + \gamma_3 \xi^3 = 0. \quad (15)$$

One can easily prove that the solution $\xi(t)$ has a frequency ω which depends on the amplitude A through

$$\omega = \Omega_0 \left(1 - \frac{5\gamma_2^2 A^2}{12\Omega_0^4} + \frac{3\gamma_3 A^2}{8\Omega_0^2} \right). \quad (16)$$

By comparing (14) and (15) one identifies

$$\Omega_0^2 = 2\omega_\perp^2 \left(1 - A_2^2 \frac{16 - 5\lambda^2}{16 - 7\lambda^2} \right) \quad (17)$$

$$\gamma_3 = 2\omega_\perp^2 \left(1 + \frac{16 - 3\lambda^2}{16 - 7\lambda^2} \right) \quad (18)$$

and $\gamma_2 = 0$. Inserting these definitions into (16), one gets finally

$$\omega = \sqrt{2} \omega_{\perp} \left[1 + \frac{(16 - 5\lambda^2)}{4(16 - 7\lambda^2)} A_2^2 \right], \quad (19)$$

which is the analytic expression of the frequency shift as a function of the anisotropy of the trap. Here the pathology at $\lambda = \sqrt{16/7}$ is evident: as already said, it corresponds to a resonance induced by harmonic generation.

In a more general form, one can write

$$\omega = \omega_0 [1 + \delta(\lambda) A^2], \quad (20)$$

where ω_0 is the frequency of each normal mode in the linear regime ($\sqrt{2}\omega_{\perp}$ and $\sqrt{q_{\mp}}\omega_{\perp}$ for the $m = 2$ and $m = 0$ modes, respectively) while A is its amplitude. The coefficient for the $m = 2$ mode is then

$$\delta_2(\lambda) = \frac{(16 - 5\lambda^2)}{4(16 - 7\lambda^2)}. \quad (21)$$

Expansion (20) is expected to be reliable whenever $|\delta|A^2 \ll 1$ and this excludes values of λ too close to resonances. The coefficients $\delta_{\mp}(\lambda)$ for the $m = 0$ modes can be calculated straightforwardly, with the same procedure used above for $m = 2$. In particular, one has to start exciting one of the two modes, ξ_{\mp} , in the form $A_{\mp} \cos(\sqrt{q_{\mp}} \omega_{\perp} t)$; the other one is excited to the order A_{\mp}^2 , while ξ_2 is never excited, due to the axial symmetry of both the equations and the initial conditions. Expanding (6) up to terms in A_{\mp}^3 , one gets two coupled equations for ξ_{\mp} of the type (15), while ξ_2 is decoupled. One finally obtains

$$\begin{aligned} \delta_{\mp} = & \frac{5}{2} \lambda^2 \frac{(q_{\pm} - 2)(q_{\mp} - 4)(q_{\mp} - 5)}{(4q_{\mp} - q_{\pm})(q_{\pm} - q_{\mp})^2} \left[-1 + \frac{15}{4} \frac{\lambda^2}{q_{\mp}^2} \right] \\ & - \frac{15}{16} \frac{1}{(q_{\pm} - q_{\mp})^2} [-q_{\mp} + 2\lambda^2 q_{\mp} - 9\lambda^2 + 8]^2 \\ & - \frac{9}{4} \frac{(q_{\pm} - 4)}{q_{\mp}(q_{\mp} - q_{\pm})} \\ & - \frac{3}{20} \frac{q_{\mp} - 3}{q_{\mp} - q_{\pm}} [-10\lambda^2 q_{\mp} + 37\lambda^2 + 11q_{\mp} - 54]. \quad (22) \end{aligned}$$

Two resonances are found by exciting the low-lying $m = 0$ mode. They occur when the frequency of the high-lying mode is equal to the second harmonic of the low-lying mode; this happens for $\lambda = (\sqrt{125} \pm \sqrt{29})/\sqrt{72}$ (i.e., $\lambda \approx 0.683$ and $\lambda \approx 1.952$). Conversely, no resonances are found by exciting the high-lying $m = 0$ mode. As an example of numerical values of δ_{\mp} , let us consider a spherical trap, $\lambda = 1$; in this case one has $q_- = 2$, $q_+ = 5$, $\delta_- = 11/12$ and $\delta_+ = -1/6$. The latter value was already used in Ref. [25] to give an estimate of the collapse time for the high lying $m = 0$ mode.

The main results of these analytic expansions, together with the numerical solutions of (6), will be discussed in the following section.

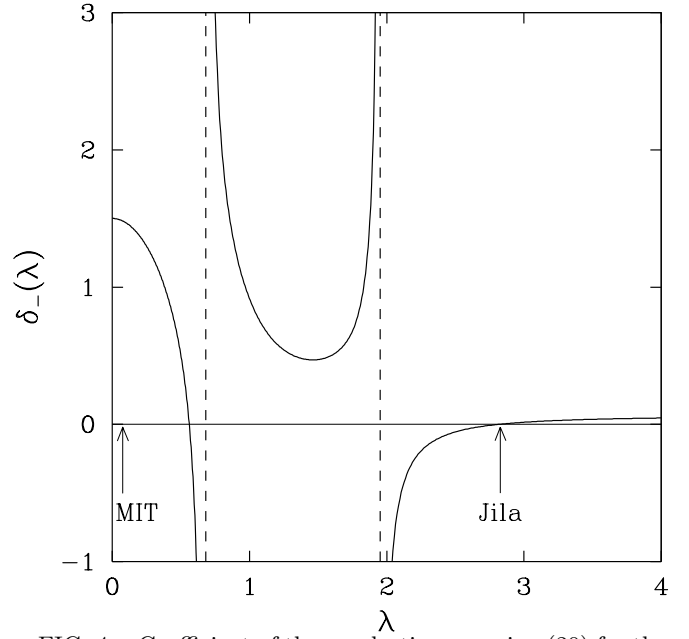


FIG. 4. Coefficient of the quadratic expansion (20) for the frequency shift of the low-lying $m = 0$ mode, as a function of the anisotropy parameter $\lambda = \omega_{0z}/\omega_{\perp}$. Divergences are found for $\lambda = (\sqrt{125} \pm \sqrt{29})/\sqrt{72}$. The values of λ for the experimental traps of Refs. [1,2] are also indicated.

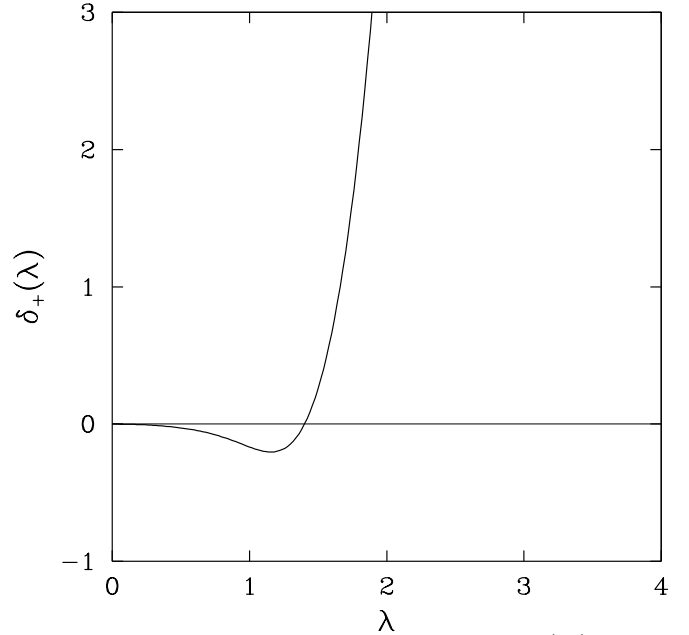


FIG. 5. Coefficient of the quadratic expansion (20) for the frequency shift of the high-lying $m = 0$ mode. For $\lambda = \sqrt{8}$ the coefficient is about 31.

V. FREQUENCY SHIFT AND MODE COUPLING FOR DIFFERENT TRAPS

So far we have developed two methods for investigating the behavior of the $m = 0$ and $m = 2$ modes in a cylindrically symmetric trap. One is the numerical solution of the coupled differential equations (6) for the widths $b_j(t)$, which corresponds to solve exactly the equations of hydrodynamics for these collective modes. The other one is an expansion at small amplitude, which provides analytic solutions as well as simple formulae for the frequency shift.

In Figs. 4-6 we first show the three curves for the coefficients $\delta_-(\lambda)$, $\delta_+(\lambda)$ and $\delta_2(\lambda)$, from (21-22), entering the quadratic expansion (20) of the frequency shift. These curves can be considered the main results of the present work. The occurrence of resonances is a peculiar feature of the dynamics of the condensate. Close to these special values of λ where $\delta(\lambda)$ diverges, the frequency shift is expected to show an anomalous enhancement and the time evolution of the widths of the condensate should become very irregular, due to significant mode coupling.

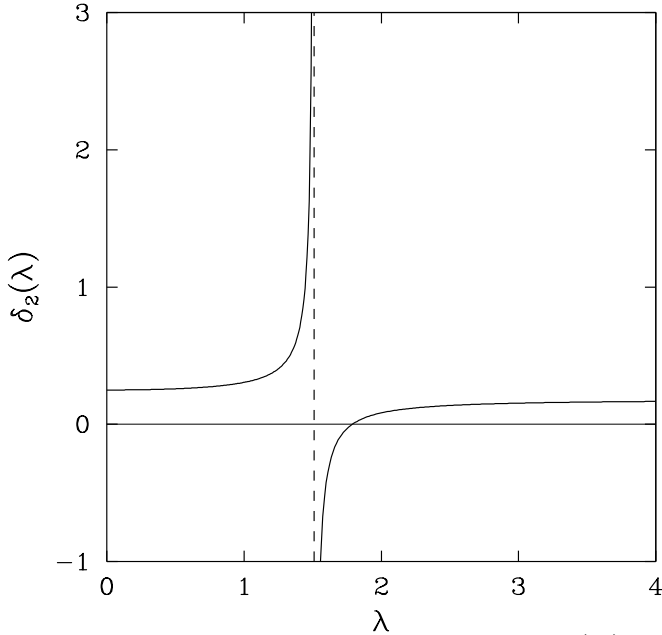


FIG. 6. Coefficients of the quadratic expansion (20) for the frequency shift of the $m = 2$ mode. A divergence is found at $\lambda = \sqrt{16/7}$.

In order to test the accuracy of the small amplitude expansion, yielding the curves in Figs. 4-6, we compare the results obtained from both the numerical solution of (6) and the quadratic law (20) for specific values of λ . In Fig. 7 we show the results for the $m = 2$ and low-lying $m = 0$ modes at $\lambda = \sqrt{8}$, as in Jila experiments [1]. In this case one finds the coefficients $\delta_2(\sqrt{8}) = 3/20$ and $\delta_-(\sqrt{8}) = 1.636 \times 10^{-3}$. Solid lines are obtained from the numerical solutions of (6), using definitions (7-

9), and performing sinusoidal fits and/or Fourier analysis of ξ_2 and ξ_- . The frequency shift of the $m = 0$ mode is smaller than the one of $m = 2$ especially for small amplitudes; in the $A \rightarrow 0$ limit, the agreement between the numerical results and the quadratic law (dashed lines) is very good. In the $m = 2$ case it remains good even for large amplitudes, while higher order corrections seem to be important for the $m = 0$ mode at amplitudes larger than 0.2 (corresponding to relative amplitudes of the order of 20% in the radial width of the condensate). In the latter case, one notices also that the coefficient δ_- is very small, since the value $\lambda = \sqrt{8}$ is incidentally very close to a root of the function $\delta_-(\lambda)$, as shown in Fig. 4. Thus the first nonzero contributions to the shift come from powers A^3 or higher. We note also that the abscissa in Fig. 7 is the amplitude of $\xi_2(t)$ and $\xi_-(t)$ and not of the width $b_x(t)$, which is the observed quantity at Jila. The reason is that the quadratic expansion (20) have analytic coefficients only for the normal modes ξ_j . However, since $\lambda = \sqrt{8}$ is relatively far from resonances, the coupling between different modes in (7-8) is rather weak and the relative amplitude of the oscillations of b_x practically coincides with the amplitude of each normal mode, within the range of Fig. 7.

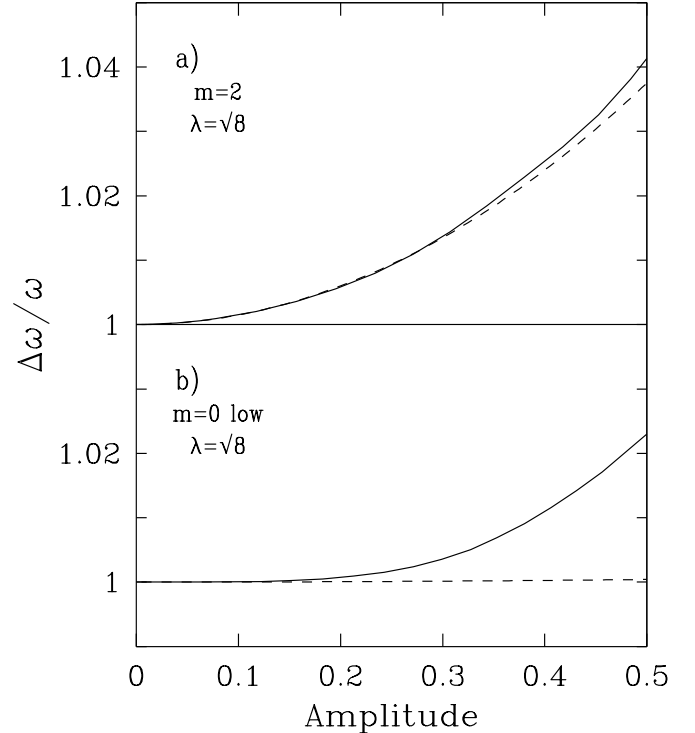


FIG. 7. Frequency shift for the $m = 2$ and low-lying $m = 0$ modes in the Jila trap as a function of their amplitude. Solid line: from the numerical solution of (6); dashed line: quadratic expansion (20) with $\delta_2 = 3/20$ and $\delta_- = 1.636 \times 10^{-3}$ from (21-22).

It is not easy to compare these results with the available data for the frequency shift in the Jila experiments

[1]. The data show no significant frequency shift for the low-lying $m = 0$ mode and a relatively large shift for the $m = 2$ mode. The fact that the shift is larger for $m = 2$ than for $m = 0$ is in agreement with our predictions, but a quantitative comparison should take into account at least two further effects. First, the data have been taken after switching off the trapping potential. As discussed in our previous work [17] (see also [23]), the expansion of the gas causes an amplification of the relative amplitude of the observed oscillations. This amplification is different in the three directions, depending on the anisotropy of the trap and on the excited modes. Second, the assumption $Na/a_{HO} \gg 1$, which is at the basis of the hydrodynamic approach, is not well satisfied by the samples used so far at Jila. In this case, the role of the quantum pressure, beyond the Thomas-Fermi approximation could be relevant also for the dynamics of the expansion.

In Fig. 8 we plot the frequency shift of the low-lying $m = 0$ mode for the MIT trap [2] with $\lambda = 0.077$. Again the solid line comes from the evolution of $\xi_-(t)$, calculated numerically from (6), while the dashed line is the quadratic approximation (20), whose coefficient is now $\delta_- = 1.481$. The agreement between the two curves is good. In this trap, the condensate is cigar-shaped and one measures the oscillations of the axial width $b_z(t)$. Looking at equation (9), one notes that the relative amplitude of the oscillations of b_z is about four times the amplitude of the normal mode ξ_- . In fact, for such a small value of λ , the quantity q_- behaves like $(5/2)\lambda^2$ and, hence, can be neglected in (9); moreover, the coupling with the high-lying $m = 0$ mode is weak and thus one can also neglect ξ_+ at the lowest order. This means that the shift $\delta_- A^2$ becomes $(\delta_-/16)A_z^2$, if A_z is the relative amplitude of the axial width b_z . The shift in frequency is thus very small; for instance, when the axial width oscillates with a relative amplitude of 40%, the amplitude of ξ_- is about 0.1 and the predicted shift is less than 2%. Actually the oscillations in Ref. [2], which do not exhibit any frequency shift, have been imaged after the free expansion of the condensate and correspond to relative amplitudes of the order of 10%, or less, for b_z before the expansion. Therefore, the predicted shift is practically zero within the accuracy of the available experiments and the A^2 law can not be tested. The situation can be greatly improved by the use of nondestructive imaging techniques, as in the trap of Ref. [24].

As concerns nonlinear coupling of $m = 0$ and $m = 2$ modes, the values of λ in the Jila and MIT traps are not of particular interest. The curves in Figs. 4-6 suggest different choices, namely $\lambda = \sqrt{16/7}$ for the $m = 2$ mode and $\lambda = (\sqrt{125} \pm \sqrt{29})/\sqrt{72}$ for the low-lying $m = 0$ mode. It is also worth mentioning that the curves in Figs. 4-6 come from an expansion up to the third order in A , as in (11-12), and the coupling occurs *via* the second harmonics. When the amplitude is large, one expects higher order terms to become significant, providing other special values of λ for mode coupling. For instance,

the hydrodynamic equations (6) predict a beating of the $m = 2$ and the high-lying $m = 0$ modes for $\lambda = \sqrt{63/11}$. This beating can be calculated analytically by extending equations (11-12) to the next order in A ; for that value of λ , one then finds that the third harmonic of ξ_2 has the same frequency of ξ_+ . Finally, we remark that other values of λ can give rise to similar effects, even without harmonic generation. This happens when an accidental degeneracy occurs in the spectrum of normal modes, including $|m| > 2$, as recently discussed by Öhberg *et al.* [19].

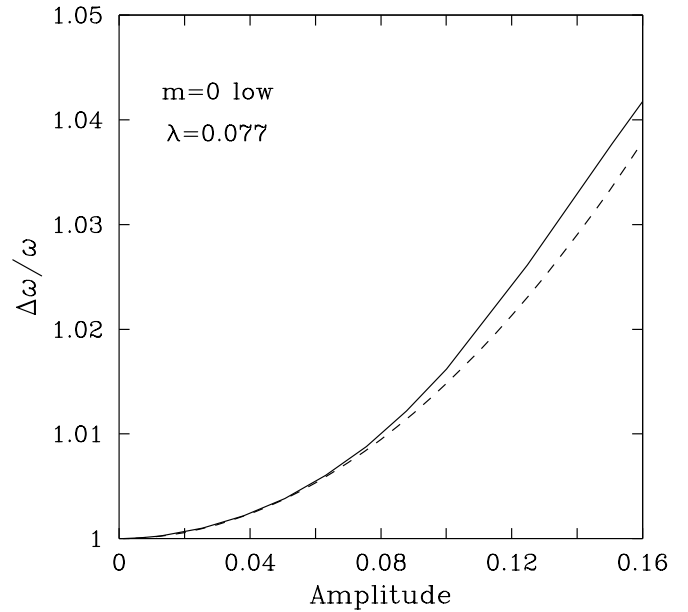


FIG. 8. Frequency shift for the low-lying $m = 0$ mode in the MIT trap as a function of its amplitude. Solid line: from the numerical solution of (6); dashed line: quadratic expansion (20) with $\delta_- = 1.481$ from (22).

VI. APPLICATION TO THE THEORY OF COLLAPSE AND REVIVAL

The fact that the frequency of the normal modes depends on their amplitude has interesting consequences for the phenomenon of collapse and revival of collective excitations. The latter belongs to a class of quantum phenomena having no classical analogue. Examples are the collapse and revival of coherent quantum states observed in atomic Rydberg wave packets [26], in molecular vibrations [27] and for atoms interacting with an electromagnetic field in a resonant cavity [28] (see [29,30] and references therein, for a theoretical discussion). The same concepts have been recently applied to the trapped condensate (see refs. [25,32] and references therein). In this confined Bose system, the collapse of collective excitations originates from a dephasing during the oscillation. In Ref. [32] the dephasing was associated with the fluctuations of the particle occupation number and the collapse

time was estimated within Hartree approximation. One of us [25] has recently developed a theory for the same process, but based on the fluctuations in the number of quanta of oscillation. The main idea is that a collective mode is, in general, a coherent superposition of stationary states of an oscillator and the number of quanta of oscillations is the natural quantity for classifying those states. The estimated time scale for the collapse turns out to depend, as a key ingredient, on the frequency shift of the normal modes. We will recall here the main steps of the theory and then we will use our new results for the frequency shifts in order to provide a quantitative estimate of the collapse time.

As we saw in section IV, the collective variable ξ can be considered as solutions of the equations for anharmonic oscillators. Let us write the frequency, up to the first non linear correction, in the form

$$\omega = \omega_0 + \delta\omega = \omega_0(1 + \kappa E), \quad (23)$$

where E is the energy of the oscillations. We assume here the condition of weak nonlinearity, $|\kappa|E \ll 1$, even though the theory can be generalized to the case of large amplitude. In order to provide a quantum description, one can use the semiclassical expression $\hbar\omega = (\partial E_n / \partial n)$, which allows one to rewrite equation (23) in the quantum form $\omega_n = E_n / \hbar = \omega_0 n + bn^2/2$. Here the quantity n is the number of quanta ($n \gg 1$) in a given excited state of energy E_n , while the coefficient b is given by $b = \hbar\omega_0^2\kappa$. The oscillations in the experiments [1,2] were driven through a sinusoidal force into a coherent state of the oscillator. The wave function of a such a stationary state can be written in the form

$$\psi = \sum_n c_n \psi_n \exp(-i\omega_n t), \quad (24)$$

where

$$|c_n|^2 = \frac{\bar{n}^n}{n!} \exp(-\bar{n}) \approx \frac{1}{\sqrt{2\pi\bar{n}}} \exp\left[-\frac{(n - \bar{n})^2}{2\bar{n}}\right], \quad (25)$$

where \bar{n} is the average number of quanta ($\bar{n} \gg 1$) and $E = \hbar\omega_0\bar{n}$. Let us now consider the oscillator co-ordinate $\xi(t)$ and calculate its average over the state (24-25). If one considers only $n \rightarrow n \pm 1$ transitions, the result is $\langle \xi(t) \rangle \propto \sum_n |c_n|^2 \cos[(\omega_0 + bn)t]$. For small enough values of t , one can replace the summation over n by an integral and one gets a Gaussian damping of the oscillation according to $\langle \xi \rangle \sim \exp(-\bar{n}b^2t^2/2) \equiv \exp[-(t/\tau_c)^2]$, where

$$\tau_c^{-1} = (\bar{n}/2)^{1/2} |b| = \omega_0(E\hbar\omega_0/2)^{1/2} |\kappa|. \quad (26)$$

Since the amplitude of the oscillations is proportional to $\sqrt{\bar{n}}$, the resulting amplitude dependence of the collapse time, τ_c , is the same as in the theory of Ref. [32]. The periodicity of $\langle \xi(t) \rangle$ gives also the revival period $\tau_r = 2\pi/(\hbar\omega_0^2|\kappa|)$ as in Ref. [29]. The meaning of τ_c and τ_r is also shown schematically in Fig. 9. From the expressions

for τ_c and τ_r one easily sees that $\tau_c \approx \sqrt{(1/\bar{n})} \tau_r \ll \tau_r$. Note that in quantum mechanics the measurement of an oscillator coordinate ξ is, generally speaking, destructive because it affects the oscillator momentum p_ξ . In other words, it modifies the coefficients c_n , thus preventing the observation of collapse and revival. In order to measure properly $\langle \xi(t) \rangle$, one must repeat cycles of observations with different replica of the system in different instants of time. This is, however, exactly the procedure used in the available experiments [1,2,31].

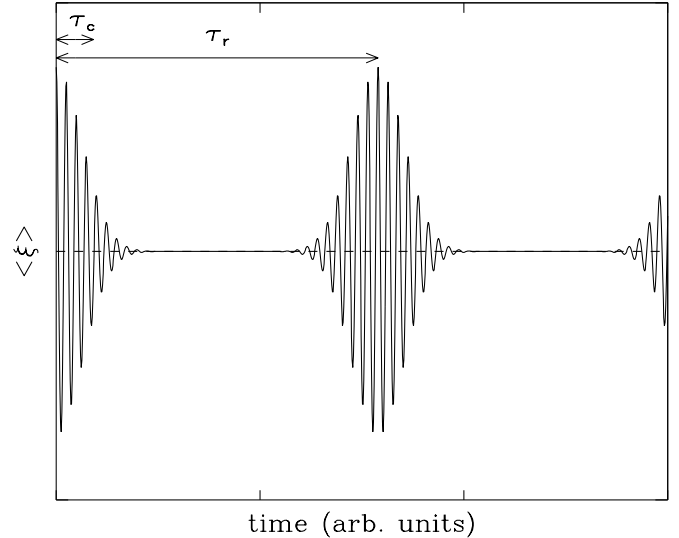


FIG. 9. Schematic picture of collapse and revival. Both time and $\langle \xi \rangle$ are in arbitrary units and the oscillations are just a pictorial view of the phenomenon. The two time-scales, τ_c and τ_r are indicated.

According to (26) the collapse time τ_c decreases when \bar{n} increases. However, one has to keep in mind that the theory is restricted to the weak nonlinear regime $|\kappa|E \ll 1$, which corresponds to $\bar{n} \ll 1/(\hbar\omega_0|\kappa|)$. Thus, the collapse time is subject to the condition $\tau_c \gg \tau_{min} \equiv 1/(\omega_0\sqrt{\hbar\omega_0|\kappa|})$ and the collapse can be observed only if τ_{min} is shorter than the typical time scale for the dissipative damping of the oscillations, which is always present at finite temperature.

In order to provide quantitative estimates of the collapse time for trapped Bose condensed gases, one must calculate the energy of oscillations, E , as a function of their amplitude. This is what we have done in the previous sections. One has only to write the proper relation between the nonlinearity coefficients κ and δ . To this aim, it is enough to calculate the energy E in linear approximation, as twice the mean kinetic energy $E = m \int d\mathbf{r} \rho_0 v^2$, where $\rho_0(\mathbf{r})$ is the equilibrium density of condensate and $\mathbf{v}(\mathbf{r})$ is the velocity field associated with each normal mode. Direct integration for $m = 2$ gives:

$$E = \frac{4}{7} \mu N A_2^2, \quad (27)$$

where A_2 is the amplitude of the mode and μ is the

chemical potential of the system, calculated within the Thomas-Fermi approximation. Similarly, for the $m = 0$ modes one finds

$$E = \frac{1}{7} [2q_{\mp} + (q_{\mp} - 4)(3q_{\mp} - 10)] \mu N A_{\mp}^2, \quad (28)$$

where the quantities $q_{\mp}(\lambda)$ are defined in (10). The above energies can be rewritten in a more compact form as $E = \epsilon \mu A^2$, where $\epsilon(\lambda)$ is the appropriate coefficient for each normal mode in (27-28). Inserting this result in (23) and comparing with the quadratic law (20), one finds

$$\kappa = \frac{\delta}{\epsilon \mu N}. \quad (29)$$

The final result for the collapse time, from (26), is then

$$\omega_0 \tau_c = \frac{\sqrt{2} \epsilon \mu N}{|\delta| \sqrt{E \hbar \omega_0}} = \frac{\sqrt{2} \epsilon}{A |\delta|} \sqrt{\frac{\mu N}{\hbar \omega_0}}. \quad (30)$$

Let us estimate τ_c for the $m = 2$ mode, assuming typical experimental parameters of Ref. [1]: $N = 4500$, $\omega_{\perp}/2\pi = 132$ Hz, $\lambda = \sqrt{8}$ and $a = 110$ Bohr radii. With these parameters one finds $\sqrt{\mu N/\hbar \omega_{\perp}} \approx 195$. The frequency of the $m = 2$ mode in the linear limit is $\omega_0 = \sqrt{2} \omega_{\perp}$, while the coefficient of the quadratic expansion (20) is $\delta_2 = 3/20$. Equation (27) gives $\epsilon_2 = 4/7$. With a reasonable choice for the amplitude, $A = 0.2$, the final result for the collapse time is $\tau_c = 4.98$ s. The authors of Refs. [1,31] reported a lifetime of the $m = 2$ mode of the order of 100 ms. It means that, under the experimental conditions, the dissipative damping is too strong and the collapse can not be observed. It is not hopeless, however, to discover this effect under appropriate conditions. The point is that, according both to measurements [31] and to theoretical considerations [33], the dissipative damping decreases rapidly by lowering the temperature of the gas. We believe that experiments at lower temperature would permit to observe the quantum collapse of collective modes in these macroscopic objects. It is worth mentioning, in this connection, the recent suggestion to cool adiabatically the gas by changing the trapping frequency [34].

The low-lying $m = 0$ mode, which also has been observed, exhibits a very small frequency shift and is not, for this reason, a proper object to search for collapse, at least within the range of amplitude where the quadratic expansion (20) holds. Conversely, the high-lying $m = 0$ mode looks more promising. For this mode, using again the parameters of the Jila trap, one has $\omega_0 \approx 4.98 \omega_{\perp}$, $\epsilon \approx 198$ and $\delta_{+} \approx 30$. The oscillations are strongly anisotropic and the amplitude in the z -direction is larger than in the radial one. According to (9) one gets $A_z \approx (q_{+} - 4)A_{+} \approx 20.8A_{+}$. By assuming again $A_z = 0.2$, the collapse time becomes $\tau_c = 1.4$ s.

From these results, it appears evident that the *resonances* described in the previous sections of this work are very promising even for the observation of collapse

and revival of the oscillations. By a proper choice of the anisotropy parameter λ one can in fact increase significantly the coefficient δ and, consequently, lower the collapse time to an observable scale. Obviously, for real traps, one should also take into account possible nonlinear effects originating from trivial nonharmonic corrections to the magnetic confining potential. But the present scenario for collapse and revival seems plausible and, in any case, the first step for its confirmation should be the observation of the predicted amplitude dependence of the frequency shift.

VII. CONCLUSIONS

In this paper we have investigated the behavior of collective excitations of trapped Bose gases in the limit of zero temperature. The formalism of hydrodynamic equations is suitable for deriving both numerical and analytic results, valid in the limit $Na/a_{HO} \gg 1$. As already shown by other authors, those equations describe the lowest collective excitations in the linear regime (normal modes), in agreement with available experiments. They can be also used for the dynamics of the system in nonlinear regime. We have shown here that they provide non-trivial predictions for the frequency shift of the normal modes. We have discussed the case of $m = 0$ and $m = 2$ modes in a generic cylindrically symmetric trap and we have explored the peculiar behavior of the condensate for special choices of $\lambda = \omega_{0z}/\omega_{\perp}$, the asymmetry parameter of the trap. For those special traps, the frequency of an oscillation becomes equal to the second harmonic of another one. This degeneracy enhances nonlinear effects, producing significant consequences in measurable quantities: for instance, the time evolution of condensate shape can show irregular patterns and the frequency shift can become rather large. The physical picture has been supported by simple analytic results. We have also discussed an application to the theory of collapse and revival of the collective oscillations.

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